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# Adaptive law with a new leakage term

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**Abstract:** The hyperstability theory of adaptive control systems is extended to encompass plants with unmodelled dynamics and disturbances. The analysis not only shows that leakage in the adaptive law is a natural way to avoid robustness problems (which is a known result in adaptive control theory), but also provides a new adaptive law that is a sort of signal-dependent  $\sigma$ -modification. The proposed adaptive law is less conservative than the  $\sigma$ -modification, but still ensures the global stability of the system, which is formally proven in the paper. Since it is shown that the design parameter  $\sigma'$  of the proposed adaptive law is directly related to the  $H_\infty$  norm of the parasitic dynamics, the criteria for system stability are derived. Based on these, some guidelines for choosing the leakage parameter  $\sigma'$  and the bandwidth of the reference model are presented.

## 1 Introduction

At the end of the 1970s much effort was put into proving the stability of adaptive algorithms. Several stability proofs were derived showing the stability of the adaptive system based on certain assumptions (knowledge of the plant order, no disturbances, noises etc.). It was shown that a relatively slight violation of these assumptions can result in system instability [1]. In the 1980s there was a lot of work on robust adaptive control in order to try to minimise the negative influence of parasitics and disturbances in the system [2]. One of the best known approaches to robust adaptive control is the use of leakage in the adaptive law. Well-known types of leakage are  $\sigma$ -modification [3]  $e_1$ -modification [4], switching  $\sigma$ -modification [5] etc. The general idea of the leakage is developed further in [6], where the leakage parameter switches to some signal-dependent term when the estimates are far from the a priori guess, and in [7], where a so-called 'circular-leaky' algorithm is introduced. Another possibility for preventing parameter bias is the use of directional leakage [8], where the leakage is applied only in the directions in parameter space in which the exciting signal is not informative. Leakage is sometimes also referred to as soft projection. Similar results can be obtained by the use of parameter projection [9].

To prove the stability of the classical adaptive systems several mathematical tools were used such as Lyapunov functions and Popov hyperstability theory [10]. In this

paper the latter will be extended to the case where there are parasitics and disturbances present in the system. The analysis will show the very well-known fact that the inclusion of leakage into the adaptive law is a natural way to stabilise the system in the presence of parasitics. This analysis will also lead to a new adaptive law with leakage, which is similar to the  $\sigma$ -modification but less conservative and more robust to different amplitudes of the signals in the system. A very important property of the proposed adaptive law is that the leakage parameter is not dependent on any signals present in the system.

One of the main drawbacks of adaptive control algorithms is the lack of guidelines for choosing the design parameters in such systems. Some issues on tuning the  $\sigma$  parameter have been discussed by [11], especially those that influence the onset of bursting. Another route was taken by [12]; they proposed an algorithm for adjusting the  $\sigma$  parameter on line, based on the a posteriori errors obtained using the algorithm with and without leakage. One of the important results of the analysis carried out in this paper is the fact that some criteria on ensuring the stability of the system in the presence of parasitic dynamics will be obtained. Consequently, some guidelines on choosing the design parameters in direct model reference adaptive control (MRAC) systems will be obtained.

In Section 2 some preliminaries are given. In Section 3 MRAC systems with parasitics are analysed. Section 4

introduces novel adaptive law whereas some guidelines for choosing design parameters are given in Section 5. The conclusions are stated in Section 6.

## 2 Problem formulation

The goal of this paper is to adaptively control the unknown minimum phase plant of arbitrary order ( $n^*$ ) and arbitrary relative degree ( $n^* - m^*$ )

$$y_p = \frac{B^*(s)}{A^*(s)}(u + d_u) = \frac{\kappa^* \prod_{i=1}^{m^*} (s - \zeta_i^*)}{\prod_{i=1}^{n^*} (s - \pi_i^*)} (u + d_u) \quad (1)$$

where  $u$  is the control input and  $y_p$  is the controlled output. Since the plant is linear and minimum phase, all the bounded disturbances in the plant can be replaced by one bounded disturbance at the plant input – denoted by  $d_u$ . The poles  $\pi_i^*$  and the zeros  $\zeta_i^*$  are indexed in ascending order of the absolute value. The polynomials  $A(s)$  and  $B(s)$  are defined as

$$A(s) = \prod_{i=1}^n (s - \pi_i^*) = s^n + a_1 s^{n-1} + \dots + a_n \quad (2)$$

$$B(s) = \kappa \prod_{i=1}^m (s - \zeta_i^*) = b_0 s^m + b_1 s^{m-1} + \dots + b_m \quad (3)$$

where  $\kappa$  is selected to meet the zero-frequency condition

$$\lim_{s \rightarrow 0} \frac{B(s)}{A(s)} = \lim_{s \rightarrow 0} \frac{B^*(s)}{A^*(s)} \quad (4)$$

The transfer function  $B(s)/A(s)$  therefore represents the low-frequency approximation of  $B^*(s)/A^*(s)$ . The remaining poles and zeros from  $B^*(s)/A^*(s)$  are included in  $\Delta(s)$  so that the following holds

$$\frac{B(s)}{A(s)}(1 + \Delta(s)) = \frac{B^*(s)}{A^*(s)} \quad (5)$$

The transfer function  $\Delta(s)$  is therefore the parasitic (or unmodelled) dynamics of the plant in multiplicative form. Owing (4) and (5) it holds

$$\Delta(s)|_{s=0} = 0 \quad (6)$$

*Remark 1:* Note that the meaning of parasitic dynamics in adaptive control is slightly different from that in robust linear control. In the adaptive case the plant is unknown, and only the plant structure (the parameters  $n$  and  $m$ ) has to be known in advance to derive the appropriate control algorithm. Consequently, the plant dynamics can be divided in an arbitrary manner among the nominal plant model  $B(s)/A(s)$  and the parasitic dynamics. In our case, the plant model is the best approximation (based on the above criterion) of the plant in the low-frequency region.

*Remark 2:* No special requirements about the poles or the zeros are needed. This means that they can lie anywhere in the complex plane (higher multiplicities are also allowed) as long as all the zeros lie in the open left-hand-side half-plane. It is even possible to select a nominal plant order higher than the actual one ( $n > n^*$ ). In such a case the controller that achieves the perfect tracking is not characterised by a single point in the control parameter space but with a set of points.

The desired behaviour of the closed-loop system is defined by a reference model

$$y_m = G_m(s)w = \frac{B_m(s)}{A_m(s)} w = \frac{b_{m0} s^m + \dots + b_{mm}}{s^{n_m} + \dots + a_{mn}} w \quad (7)$$

where  $w$  represents the reference.

If the plant parameters are unknown and there are no disturbances and no parasitic dynamics (the plant transfer function is  $B(s)/A(s)$ ), the problem can be solved by direct MRAC. The following control law is used

$$u = \psi^T \hat{\theta} \quad (8)$$

where

$$\psi^T = \begin{cases} \left[ \frac{\alpha^T(s)}{\Lambda(s)} u \quad \frac{\alpha^T(s)}{\Lambda(s)} y_p \quad y_p \quad w \right] \in \mathbb{R}^{2n}, & n \geq 2 \\ [y_p \quad w] \in \mathbb{R}^2, & n = 1 \end{cases}$$

$$\alpha^T(s) = [s^{n-2} \quad s^{n-3} \quad \dots \quad s \quad 1] \quad (9)$$

and  $\Lambda(s)$  is an arbitrary monic Hurwitz polynomial of degree  $n - 1$  that includes the numerator polynomial of the reference model as a factor. The control parameter vector  $\hat{\theta}$  is obtained with the adaptive law. The parameter error  $\tilde{\theta}$  is defined as

$$\tilde{\theta} = \hat{\theta} - \theta \quad (10)$$

where  $\theta$  is the true control parameter of the tuned system

$$\theta^T = [\theta_1^T \quad \theta_2^T \quad \theta_3 \quad \theta_4], \quad \theta_1, \theta_2 \in \mathbb{R}^{n-1} \quad (11)$$

Obviously, the parameter vector  $\theta$  has only two elements ( $\theta_3$  and  $\theta_4$ ) if  $n = 1$ . Therefore  $\theta_1^T \alpha(s)$  and  $\theta_2^T \alpha(s)$  can be interpreted as 0 in all the equations that follow for the case of the first-order nominal plant.

If the control law of the tuned system (with the adaptation switched off) is interpreted as a pole placement control law, the feedback controller can be obtained

$$\frac{u}{-y_p} = \frac{-(\theta_2^T \alpha(s) + \theta_3 \Lambda(s))}{\Lambda(s) - \theta_1^T \alpha(s)} \quad (12)$$

The numerator and the denominator of the above feedback control are denoted by  $Q(s)$  and  $P(s)$ , respectively

$$\begin{aligned} Q(s) &= -(\theta_2^T \alpha(s) + \theta_3 \Lambda(s)) \\ P(s) &= \Lambda(s) - \theta_1^T \alpha(s) \end{aligned} \quad (13)$$

In the case of pole placement control, the Bezout identity is obtained, which in the case of model reference control where the plant zeros are cancelled becomes [13]

$$A_m \mathcal{O} = b_0 Q + P' A \quad (14)$$

where  $P'$  is the controller denominator polynomial after cancelling the plant zeros ( $P = P'(B/b_0)$ ) and  $\mathcal{O}$  is interpreted as the observer polynomial whose roots are the roots of  $\Lambda(s)$  that are not included in  $B_m(s)$

$$B_m(s) \mathcal{O}(s) = b_{m0} \Lambda(s) \quad (15)$$

Since the direct adaptive approach is adopted here, the controllability problem is not an issue [14]. The problem arises if parasitic dynamics and disturbances are present in the system. The question is: What adaptive law should be used in combination with control law (8) in order to control the plant in a robust way?

### 3 Error model of the adaptive system

It is fairly easy to obtain the following equation that occurs often in classical adaptive literature [13]

$$\varepsilon = y_p - y_m = \frac{b_0}{b_{m0}} G_m(s) [\psi^T \tilde{\theta}] \quad (16)$$

This equation is the error model of the system. If parasitic dynamics and disturbances are present in the plant, similar error model can be obtained after some calculation

$$\begin{aligned} \varepsilon &= \overbrace{\frac{b_0}{b_{m0}} G_m \frac{1 + \Delta}{1 + \Delta(b_0 Q/A_m \mathcal{O})}}^{H_\varepsilon^0} (\psi^T \tilde{\theta}) \\ &+ \underbrace{\frac{P' A \Delta}{A_m \mathcal{O} + b_0 Q \Delta} y_m + \frac{b_0}{b_{m0}} G_m \frac{1 + \Delta}{1 + \Delta(b_0 Q/A_m \mathcal{O})}}_{d_\varepsilon} d_u \end{aligned} \quad (17)$$

where  $H_\varepsilon(s)$  is the transfer function  $\varepsilon/(\psi^T \tilde{\theta})$  and  $H_\varepsilon^0(s)$  denotes the nominal part of this transfer function (that can be seen in (16)), and  $d_\varepsilon$  is the equivalent disturbance that combines the contributions of all external signals. More compact version of the error model is

$$\varepsilon = H_\varepsilon(s) (\psi^T \tilde{\theta}) + d_\varepsilon \quad (18)$$

The error model described by (17) or (18) will be analysed. It will be particularly important if the transfer function  $H_\varepsilon(s)$  is strictly positive real (SPR) [15]. The latter demand in the nominal case requires that  $G_m(s)$  in (16) is SPR. That would limit the use of direct MRAC to plants with relative degree 1. Many adaptive schemes were designed in the late 1970s that overcome this problem and modify the design so that  $H_\varepsilon^0(s)$  is SPR even if  $G_m(s)$  is not. We assume that  $H_\varepsilon^0(s)$  is indeed SPR and that the direct MRAC applied on the plant without parasitics would result in globally stable system. Our question is: How do parasitic dynamics affect the SPR property of the error model and consequently stability of the system?

The poles of all transfer functions on the right-hand side of (17) are the same – they are equal to the closed-loop poles of the tuned system, but with unmodelled dynamics taken into account. Hence, the closed-loop poles are not exactly the same as the desired ones. If there exists the controller parametrisation  $\theta$  that stabilises the whole class of plants (1) where  $\Delta(s)$  in (5) is bounded in some way, then all transfer functions on the right-hand side of (17) are stable.

In the following the difference between the transfer functions  $H_\varepsilon(s)$  and  $H_\varepsilon^0(s)$  from (17) will be analysed. It will be denoted as  $\tilde{H}_\varepsilon^a(s)$

$$\begin{aligned} \tilde{H}_\varepsilon^a &= H_\varepsilon - H_\varepsilon^0 = H_\varepsilon^0 \frac{1 + \Delta}{1 + \Delta(b_0 Q/A_m \mathcal{O})} - H_\varepsilon^0 \\ &= H_\varepsilon^0 \frac{\Delta - \Delta(b_0 Q/A_m \mathcal{O})}{1 + \Delta(b_0 Q/A_m \mathcal{O})} = H_\varepsilon^0 \frac{P' A}{A_m \mathcal{O} + \Delta b_0 Q} \Delta \end{aligned} \quad (19)$$

where (14) is also taken into account. Our task is to show the contribution of  $\tilde{H}_\varepsilon^a$  to the violation of the SPR property, that is, we will try to estimate  $\inf_\omega \Re\{\tilde{H}_\varepsilon^a(j\omega)\}$  where  $\Re\{\cdot\}$  denotes the real part of a complex number. In order to do so, the frequency characteristics of the function in (19) will be analysed next.

Frequency response of  $\Delta(s)$ . It has already been discussed that  $(1 + \Delta)$  only includes high-frequency dynamics of the plant. Define  $\omega_{\text{par}}$  as the frequency of the dominant pole or zero in  $\Delta(s)$ . Taking into account (6), the following can be concluded

$$|\Delta(j\omega)|_{\omega \ll \omega_{\text{par}}} \ll 1 \quad (20)$$

It is very well known that it is much easier to construct adaptive controller when the relative degree (the difference between the number of poles and zeros) is small. This is why in all practical cases of designing an adaptive controller there are more poles in  $(1 + \Delta(s))$  than there are zeros, or at least their number is equal. Consequently, the frequency response  $|\Delta(j\omega)| \rightarrow K < \infty$  as  $\omega \rightarrow \infty$ . Merging the low-frequency and the high-frequency characteristics,  $\Delta(s)$

has similar frequency response like the high-pass filter with the cut-off frequency  $\omega_{par}$ .

Frequency response of  $P'A/(A_mO + \Delta b_0Q)$ . This transfer function can be rewritten as follows

$$\begin{aligned} & \frac{P'A}{A_mO + \Delta b_0Q} \\ &= \frac{P'A}{P'A + b_0Q + \Delta b_0Q} = \frac{1}{1 + (b_0Q/P'A)(1 + \Delta)} \\ &= \frac{1}{1 + (Q/P)(B/A)(1 + \Delta)} = S \end{aligned} \quad (21)$$

Equation (21) gives (output) sensitivity function  $S(s)$  of the closed-loop system with plant  $(B/A)(1 + \Delta)$  and controller  $Q(s)/P(s)$  where the latter was obtained based on the nominal plant model. The sensitivity function  $S(s)$  approaches 1 at high frequencies (cut-off frequency is near the  $G_m$  bandwidth  $\omega_{ref}$ ).

Frequency response of  $H_e^0$ . It can easily be shown that this transfer function is equal to the following

$$H_e^0(s) = T_0(s) = 1 - S(s)|_{\Delta(s)=0} \quad (22)$$

where  $T_0(s)$  is the inverse sensitivity function of the system under the assumption that the plant is without parasitics and the controller is also the nominal one  $Q(s)/P(s)$ . As such,  $H_e^0(s)$  acts as a low-pass filter with cut-off frequency near  $\omega_{ref}$ .

By introducing (21) and (22) into (19) we obtain

$$\tilde{H}_e^a(s) = T_0(s)S(s)\Delta(s) \quad (23)$$

The analysis in this section has shown that  $T_0(s)$  (low-pass filter with cut-off frequency near  $\omega_{ref}$ ) significantly defines the behaviour of  $\tilde{H}_e^a(s)$  at high frequencies whereas  $\Delta(s)$  (high-pass filter with cut-off frequency near  $\omega_{par}$ ) is the most important for the behaviour at low frequencies. Based on the relation between the two cut-off frequencies, two possible approximations of  $|\tilde{H}_e^a(j\omega)|$  are shown in Fig. 1.

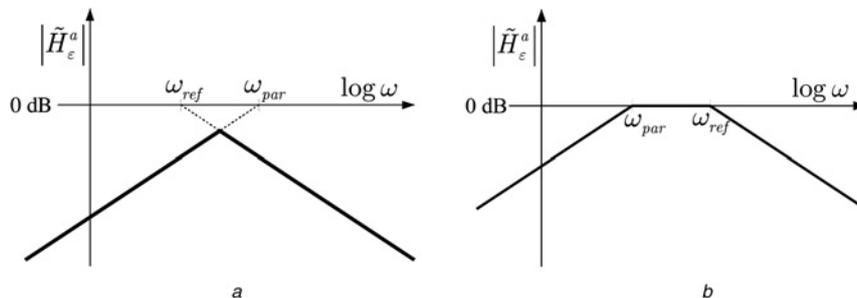


Figure 1 Approximate frequency response of  $|\tilde{H}_e^a(j\omega)|$

- a  $\omega_{par} > \omega_{ref}$
- b  $\omega_{ref} > \omega_{par}$

Note that the gain in the flat area of the frequency response in Fig. 1b is around 1 (0 dB).

If the poles play the dominant role in the parasitic dynamics (which is the usual case), the majority of the polar diagram of the  $\tilde{H}_e^a(j\omega)$  resides in the left-hand side half-plane. We are particularly interested in the point that is the farthest to the left in the polar diagram. This usually happens around phase shift of  $-180^\circ$ , which nearly coincides with the point that is the farthest from the origin of the complex plane

$$\inf_{\omega} \Re\{\tilde{H}_e^a(j\omega)\} \simeq - \sup_{\omega} |\tilde{H}_e^a(j\omega)| \quad (24)$$

## 4 Development of the adaptive law

If the very-well-known gradient adaptive law [2]

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = -\Gamma \psi \varepsilon \quad (25)$$

with  $\Gamma = \Gamma^T > 0$  being the adaptive gain is used the scheme of the adaptive system with the derived error model (18) can be constructed. The scheme is shown in Fig. 2a.

Many proofs of adaptive system stability in the 1970s relied on the Popov hyperstability theory. A very important result states [16]: 'If the system  $H_1$  in Fig. 3 is positive real (PR),  $H_2$  is SPR and  $u_1(t) \in \mathcal{L}_2$ , then  $y_1(t) \in \mathcal{L}_2$ '. It is very well known [2, 13, 10] that the feedback operator in Fig. 2a (owing to the adaptive law) is PR. The corollary gives sufficient conditions for the so-called  $\mathcal{L}_2$ -stability, that is,  $H_e^0 + \tilde{H}_e^a$  has to be SPR and  $d_c(t) \in \mathcal{L}_2$ . Since  $H_e^0$  is always SPR by adaptive law construction, we are only interested in the properties of  $\tilde{H}_e^a$ . The latter is in general not SPR, as shown in the previous section. It is known that the adaptive system can become unstable if the SPR property is violated [1].

The main idea of our approach is to add a new constant block parallel to  $\tilde{H}_e^a$ , thus improving the stability properties of the overall system, as will be shown later. A new block

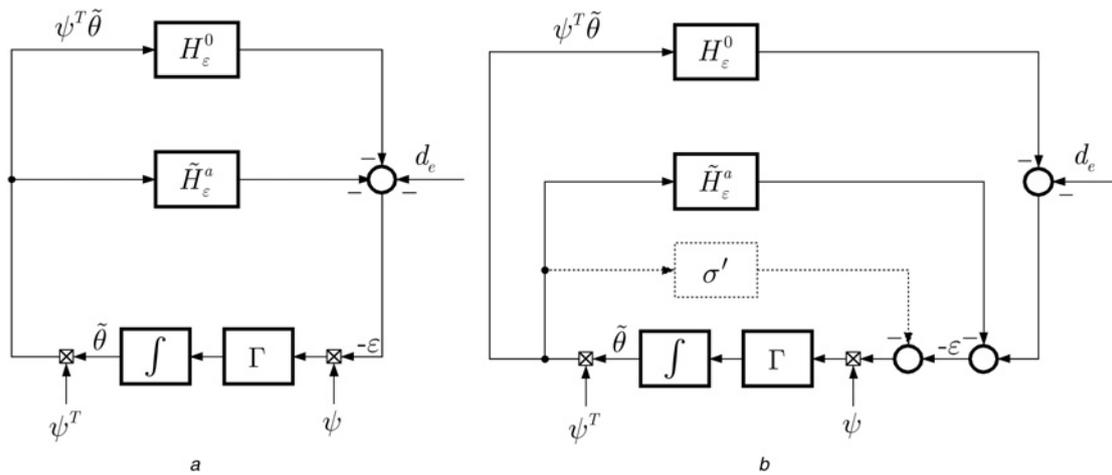


Figure 2 Scheme of the adaptive system

a Original  
b Modified

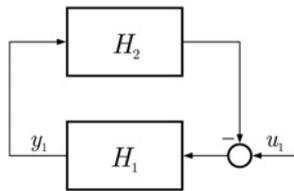


Figure 3 Feedback connection used for the stability analysis within the Popov hyperstability theory

(dotted) is added to the system, as shown in Fig. 2b. The direct operator in Fig. 2b ( $H_\epsilon^0$ ) is SPR. The feedback operator is again a feedback connection of two systems. Another result from the Popov hyperstability theory states [16]: If the systems  $H_1$  and  $H_2$  in Fig. 3 are both PR, the system with input  $u_1$  and output  $y_1$  is also PR. The feedback operator in Fig. 2b is therefore PR if  $\tilde{H}_\epsilon^a + \sigma'$  is PR, which is achieved by properly selecting the design parameter  $\sigma'$  to meet the following condition

$$\Re\{\tilde{H}_\epsilon^a(j\omega)\} \geq -\sigma', \quad \forall \omega \in \mathbb{R} \quad (26)$$

Finally, the system in Fig. 2b is hyperstable if the system in feedback is PR (this is true if condition 26 is satisfied) and  $d_e \in \mathcal{L}_2$ . The latter demand means that only disturbances of the finite energy are allowed. This is a very unrealistic requirement, and it will be shown in the following how to circumvent it.

By including  $\sigma'$  into the scheme of the adaptive system in Fig. 2b, the adaptive law has been modified as follows

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = \Gamma\psi(-\epsilon - \sigma'\psi^T\tilde{\theta}) = -\Gamma\psi\epsilon - \Gamma\sigma'\psi\psi^T\tilde{\theta} \quad (27)$$

The obvious problem of the adaptive law given by (27) is that it is not realisable as an adaptive law since it contains the

unknown  $\tilde{\theta} = \hat{\theta} - \theta$ . But we can use any possible a priori estimate for  $\theta$ , which will be denoted  $\theta^*$  (even 0 if there is no information available) to produce the following adaptive law

$$\dot{\hat{\theta}} = -\Gamma\psi\epsilon - \Gamma\sigma'\psi\psi^T(\hat{\theta} - \theta^*) \quad (28)$$

It can be seen that the adaptive law (28) uses leakage. The leakage parameter is  $\sigma'\psi\psi^T \in \mathbb{R}^{2n}$ . Such a leakage term only acts in the direction of the regressor vector  $\psi$  and is not successful in preventing parameter drift in other directions of the parameter space. This ascertainment is linked to the demand that the disturbance has to be of finite energy to ensure global stability. To overcome this problem,  $\psi\psi^T$  is substituted by its trace  $\psi^T\psi \in \mathbb{R}$  in the adaptive law, and the leakage will act in all dimensions of the parameter space

$$\dot{\hat{\theta}} = -\Gamma\psi\epsilon - \Gamma\sigma'\psi^T\psi(\hat{\theta} - \theta^*) \quad (29)$$

**Theorem 1:** Applying the control law given by (8) and the adaptive law given by (29) to the plant given by (1) results in a stable system (in the sense that all the signals in the system are bounded) provided that the following conditions have been met:

- the parasitic dynamics described by  $\Delta(s)$  in (5) are bounded so that the roots of

$$A_m O + b_0 Q \Delta = P' A + b_0 Q + \Delta b_0 Q = 0 \quad (30)$$

lie in the left-hand-side half-plane, where  $Q(s)$  and  $P(s) = P'(s)(B(s)/b_0)$  are the linear controller numerator and denominator polynomials, respectively, that enable perfect tracking when  $\Delta(s) = 0$  and  $d_u = 0$ ,

- the parameter of the adaptive law  $\sigma'$  satisfies

$$\sigma' > -\inf_{\omega \in \mathbb{R}} \Re\{\tilde{H}_\varepsilon^a(j\omega)\} \quad (31)$$

where  $\tilde{H}_\varepsilon^a$  is defined in (19),

- the equivalent disturbance  $d_c$  defined in (17) is bounded,
- there exists  $t_0 \geq 0$ , such that for each  $t \geq t_0$  the following inequality is satisfied

$$|\tilde{\theta}(t)| \geq \frac{|d_c(t)|}{(\sigma' - \underline{\sigma}')|\psi(t)|} + \frac{\sigma'|\theta - \theta^*|}{\sigma' - \underline{\sigma}'} \quad (32)$$

where

$$\underline{\sigma}' = -\inf_{\omega \in \mathbb{R}} \Re\{\tilde{H}_\varepsilon^a(j\omega)\} < \sigma' \quad (33)$$

*Proof:* The error model given by (18) can be decomposed into three parts based on (19)

$$\varepsilon = \underbrace{H_\varepsilon^0(\psi^T \tilde{\theta})}_{\varepsilon_0} + \underbrace{\tilde{H}_\varepsilon^a(\psi^T \tilde{\theta})}_{\varepsilon_a} + d_c \quad (34)$$

As discussed in Section 3, the two transfer functions  $H_\varepsilon^0(s)$  and  $\tilde{H}_\varepsilon^a(s)$  have zero-gain at high frequencies, and are therefore strictly proper. Their minimum state-space realisations are

$$\begin{aligned} \dot{x}_0 &= A_0 x_0 + b_0 \psi^T \tilde{\theta} \\ \varepsilon_0 &= c_0^T x_0 \end{aligned} \quad (35)$$

and

$$\begin{aligned} \dot{x}_a &= A_a x_a + b_a \psi^T \tilde{\theta} \\ \varepsilon_a &= c_a^T x_a \end{aligned} \quad (36)$$

The systems described by (35) and (36) are stable, as revealed in Section 3. The system (35) is also SPR, whereas the system (36) is not in general positive real. But owing to (33), the system  $(\tilde{H}_\varepsilon^a + \underline{\sigma}')$  is PR and can be rewritten in the state-space form as

$$\begin{aligned} \dot{x}_a &= A_a x_a + b_a \psi^T \tilde{\theta} \\ \varepsilon_a + \underline{\sigma}' \psi^T \tilde{\theta} &= c_a^T x_a + \underline{\sigma}' \psi^T \tilde{\theta} \end{aligned} \quad (37)$$

Applying Kalman–Yakubovich lemma on the SPR system (35) and the PR system (37) leads to the following result: For any  $L_0 = L_0^T > 0$  there exist a scalar  $\nu_0 > 0$ , matrices  $P_0 = P_0^T > 0$  and  $P_a = P_a^T > 0$ , and vectors  $q_0$  and  $q_a$  such that

$$\begin{aligned} A_0^T P_0 + P_0 A_0 &= -q_0 q_0^T - \nu_0 L_0 \\ P_0 b_0 &= c_0 \end{aligned} \quad (38)$$

and

$$\begin{aligned} A_a^T P_a + P_a A_a &= -q_a q_a^T \\ P_a b_a - c_a &= \pm q_a \sqrt{2\underline{\sigma}'} \end{aligned} \quad (39)$$

The following Lyapunov function is proposed for the stability analysis

$$V = x_0^T P_0 x_0 + x_a^T P_a x_a + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (40)$$

Its derivative with respect to time can be calculated taking into account (35), (37) and (29)

$$\begin{aligned} \dot{V} &= x_0^T A_0^T P_0 x_0 + x_0^T P_0 A_0 x_0 + 2x_0^T P_0 b_0 \psi^T \tilde{\theta} \\ &+ x_a^T A_a^T P_a x_a + x_a^T P_a A_a x_a + 2x_a^T P_a b_a \psi^T \tilde{\theta} \\ &+ 2\tilde{\theta}^T \Gamma^{-1} (-\Gamma \psi \varepsilon - \Gamma \sigma' \psi^T \psi (\tilde{\theta} - \theta^*)) \end{aligned} \quad (41)$$

The error  $\varepsilon$  can be rewritten based on (34), (35) and (36)

$$\varepsilon = \varepsilon_0 + \varepsilon_a + d_c = c_0^T x_0 + c_a^T x_a + d_c \quad (42)$$

Introducing (38), (39), (42) and (10) into (41) and cancelling the equal terms yields

$$\begin{aligned} \dot{V} &= -(x_0^T q_0)^2 - \nu_0 x_0^T L_0 x_0 - (x_a^T q_a)^2 \pm 2x_a^T q_a \sqrt{2\underline{\sigma}'} \psi^T \tilde{\theta} \\ &- 2\psi^T \tilde{\theta} d_c - 2\sigma' \psi^T \psi \tilde{\theta}^T \tilde{\theta} - 2\sigma' \psi^T \psi \tilde{\theta}^T (\theta - \theta^*) \end{aligned} \quad (43)$$

Applying Cauchy's inequality on the sixth term on the right-hand side of (43) give

$$\begin{aligned} \dot{V} &\leq -(x_0^T q_0)^2 - \nu_0 x_0^T L_0 x_0 \\ &- \left( (x_a^T q_a)^2 \mp 2(x_a^T q_a) \sqrt{2\underline{\sigma}'} (\psi^T \tilde{\theta}) + 2\underline{\sigma}' (\psi^T \tilde{\theta})^2 \right) \\ &- 2(\sigma' - \underline{\sigma}') |\psi|^2 |\tilde{\theta}|^2 - 2\psi^T \tilde{\theta} d_c - 2\sigma' \psi^T \psi \tilde{\theta}^T (\theta - \theta^*) \\ &= -(x_0^T q_0)^2 - \nu_0 x_0^T L_0 x_0 \\ &- \left( (x_a^T q_a) \mp \sqrt{2\underline{\sigma}'} (\psi^T \tilde{\theta}) \right)^2 - 2(\sigma' - \underline{\sigma}') |\psi|^2 |\tilde{\theta}|^2 \\ &- 2\psi^T \tilde{\theta} d_c - 2\sigma' \psi^T \psi \tilde{\theta}^T (\theta - \theta^*) \end{aligned} \quad (44)$$

The first three terms on the right-hand side of (44) are always negative semi-definite, the fourth term is also negative semi-definite and has to make the derivative of the Lyapunov function negative semi-definite. The only critical terms in (44) are the last two terms. Applying Cauchy's inequality to them results in

$$\begin{aligned} \dot{V} &\leq -2(\sigma' - \underline{\sigma}') |\psi|^2 |\tilde{\theta}|^2 + 2|\psi| |\tilde{\theta}| |d_c| + 2\sigma' |\psi|^2 |\tilde{\theta}| |\theta - \theta^*| \\ &= -2(\sigma' - \underline{\sigma}') |\psi|^2 |\tilde{\theta}| \left( |\tilde{\theta}| - \frac{|d_c|}{(\sigma' - \underline{\sigma}') |\psi|} - \frac{\sigma' |\theta - \theta^*|}{\sigma' - \underline{\sigma}'} \right) \end{aligned} \quad (45)$$

If there exists  $t_0 \geq 0$  such that the condition given by inequality (32) is satisfied for each  $t \geq t_0$ , then  $V(t) \leq V(t_0)$  for each  $t \geq t_0$  owing to (45). Consequently, it follows from (40) that the signals  $x_0$ ,  $x_a$  and  $\tilde{\theta}$  are bounded. The error  $\varepsilon$  is then also bounded (see (42) and the assumption on bounded  $d_c$ ). The plant output is also bounded ( $y_p = y_m + \varepsilon$ ) and due to the minimum-phase plant, the same is true for the plant input  $u$ .  $\square$

*Remark 3:* The proposed adaptive law (29) is similar to the adaptive law with the  $\sigma$ -modification

$$\dot{\hat{\theta}} = -\Gamma\psi\varepsilon - \Gamma\sigma(\hat{\theta} - \theta^*) \quad (46)$$

The proposed approach has certain advantages. When the excitation is no longer present ( $\psi = 0$ ), the adaptation stops completely, whereas in the case of adaptive law (46) only the leakage term is active – moving  $\hat{\theta}$  towards  $\theta^*$ , which results in ‘forgetting’ of the current estimates.

*Remark 4:* It is very well known that the adaptive law with the  $\sigma$ -modification in the form of (46) produces very inaccurate results if the amplitudes of the signals change drastically. One of the main reasons for this is that the ratio between the nominal adaptation and the leakage also changes drastically. If the signals (this includes  $\psi$  and  $\varepsilon$ ) change by a factor of  $\beta$ , then the first term in (46) changes by a factor of  $\beta^2$ , whereas the second term remains the same. In the case of the proposed adaptive law (29), both terms change by a factor of  $\beta^2$ , keeping the ratio constant. Most of the existing adaptive laws involving leakage do not share this property, for example, in the  $e_1$ -modification [4] the leakage term changes by a factor of  $\beta$ , the switching  $\sigma$ -modification [5] is equivalent to the  $\sigma$ -modification when leakage is activated. The exception is the adaptive law [6]. It has been known for a long time that using (static or dynamic) normalisation overcomes this problem.

The interesting fact about the proposed adaptive law without normalisation (29) is that not only is the stability of the system achieved at different amplitudes of the signals, but also the dynamic properties do not change much. This is due to a very subtle mechanism introduced through a signal-dependent leakage term ( $\psi$  is present in the leakage). One could say that in a sense the proposed adaptive law (29) combines the advantages of the adaptive law with the  $\sigma$ -modification with and without normalisation.

*Remark 5:* If at some point in time  $\psi(t)$  becomes 0, condition (31) also becomes impossible to satisfy. The problem lies in the nature of the adaptive law in the vicinity of  $\psi = 0$ . Note that the ‘classical term’ in the adaptive law (29) is linearly dependent on  $\psi$ , whereas the ‘leakage term’ is quadratically dependent on  $\psi$  and vanishes compared to the ‘classical term’ when  $\psi$  is very small. So, effectively the leakage is turned off when small signals are driving the adaptation. When  $\psi$  is exactly 0, the adaptation is also switched off and no problems are

encountered. To overcome the above-mentioned problems, an extra linear term is added to the leakage, which is treated in the next theorem (Theorem 2).

*Remark 6:* The feature that makes the fourth condition of Theorem 1 even more problematic is that it is practically almost impossible to fulfil it. The parameter error is almost never high from some time to infinity. Rather, it is high for a shorter period, which is often referred to as a burst (bursting phenomena are quite common in many adaptive schemes). During that time, the derivative of Lyapunov function is negative and the system is ‘stabilised’ during that time. Excitation is high during a burst and the parameter error decreases. Owing to parasitic dynamics and disturbances parameter error may increase again and another burst may occur after a period of time. Note that all signals are still bounded during bursts and that bursting phenomenon occurs quite rarely or even not at all. Note also that the inclusion of the proposed leakage causes that the system is not able to perform perfect tracking even in the case of the absence of disturbances and parasitic dynamics. This is a common property of leakage-based adaptive laws.

*Theorem 2:* The next adaptive law is proposed

$$\dot{\hat{\theta}} = -\Gamma\psi\varepsilon - \Gamma(\sigma'\psi^T\psi + \sigma'_2|\psi|)(\hat{\theta} - \theta^*) \quad (47)$$

It again results in a stable system where the last condition from Theorem 1 is replaced by

- there exists  $t_0 \geq 0$  such that for each  $t \geq t_0$  the following inequality is satisfied

$$|\tilde{\theta}(t)| \geq \max\left\{\frac{|d_c(t)|}{\sigma'_2} + |\theta - \theta^*|, \frac{\sigma'|\theta - \theta^*|}{\sigma' - \sigma'}\right\} \quad (48)$$

*Proof:* The proof of the theorem follows the same lines as the proof of Theorem 1. The only difference between the two adaptive laws is the extra term ( $-\Gamma\sigma'_2|\psi|(\hat{\theta} - \theta^*)$ ) used in Theorem 2. The same Lyapunov function is used. Its derivative is different from the one shown in (43). Namely, two terms are added:  $-2\sigma'_2|\psi|\tilde{\theta}^T\tilde{\theta} - 2\sigma'_2|\psi|\tilde{\theta}^T(\theta - \theta^*)$ . These two terms are also present in inequalities analogous to (44) and (45). The latter has the following form

$$\begin{aligned} \dot{V} &\leq -2(\sigma' - \sigma')|\psi|^2|\tilde{\theta}|^2 + 2|\psi||\tilde{\theta}||d_c| + 2\sigma'|\psi|^2|\tilde{\theta}||\theta - \theta^*| \\ &\quad - 2\sigma'_2|\psi||\tilde{\theta}|^2 + 2\sigma'_2|\psi||\tilde{\theta}||\theta - \theta^*| \\ &= -2(\sigma' - \sigma')|\psi|^2|\tilde{\theta}|\left(|\tilde{\theta}| - \frac{\sigma'|\theta - \theta^*|}{\sigma' - \sigma'}\right) \\ &\quad - 2\sigma'_2|\psi||\tilde{\theta}|\left(|\tilde{\theta}| - \frac{|d_c|}{\sigma'_2} - |\theta - \theta^*|\right) \end{aligned} \quad (49)$$

If inequality (48) is satisfied for each  $t \geq t_0$ , then both expressions in parentheses in (49) are positive for  $t \geq t_0$ .

Consequently,  $V(t) \leq V(t_0)$ , and the same arguments as in the proof of Theorem 1 can be stated to prove the boundedness of all the signals in the system.  $\square$

*Example 1:* Let us take the famous Rohrs' example [1] with plant  $G_p$  (the nominal part is  $G_0$ ) and reference model  $G_m$

$$G_p(s) = G_0(s)(1 + \Delta(s)) = \frac{2}{s+1} \times \frac{229}{s^2 + 30s + 229}$$

$$G_m(s) = \frac{3}{s+3} \quad (50)$$

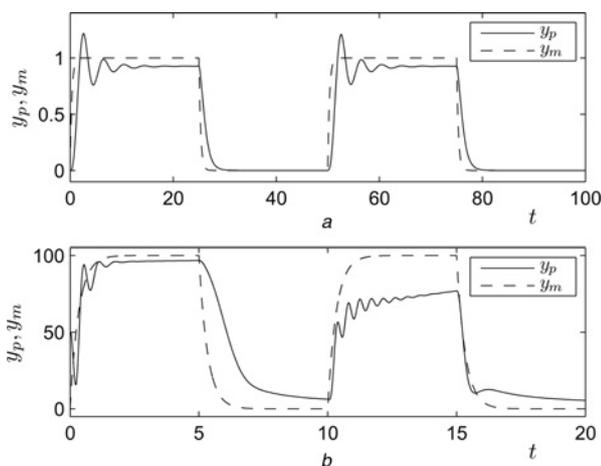
The plant operates in two environments: the simple conditions denote the case without disturbance and a periodic reference signal of pulses of amplitude 1 and the difficult conditions denote the case with output disturbance  $d_y = 50$  and a periodic reference signal of pulses of amplitude 100.

Since it is not possible to find control parameters that would result in an acceptable behaviour of the adaptive law (46) in both environments, a version with static normalisation

$$\dot{\hat{\theta}} = -\Gamma \frac{\psi}{1 + \psi^T \psi} \varepsilon - \Gamma \sigma' (\hat{\theta} - \theta^*) \quad (51)$$

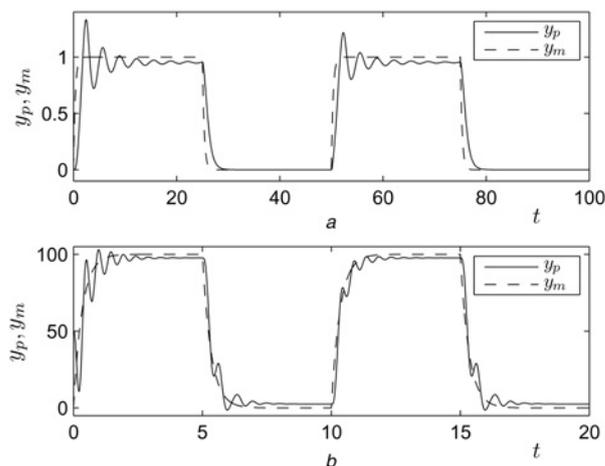
was compared with the proposed adaptive law (29). The same parameters were used in both cases:  $\Gamma = I$ ,  $\sigma = 0.1$ ,  $\sigma' = 0.1$ .

The results of the experiment with the adaptive law (46) in the simple conditions (Fig. 4a) were similar to the results with the proposed adaptive law (Fig. 5a). The performance in the difficult conditions (Fig. 4b), however, is much worse compared to the one in (Fig. 5b). This is due to the normalisation – the leakage term, being dominant, drives the controller parameters towards biased values. This effect could be reduced by increasing the adaptive gain matrix  $\Gamma$ ,



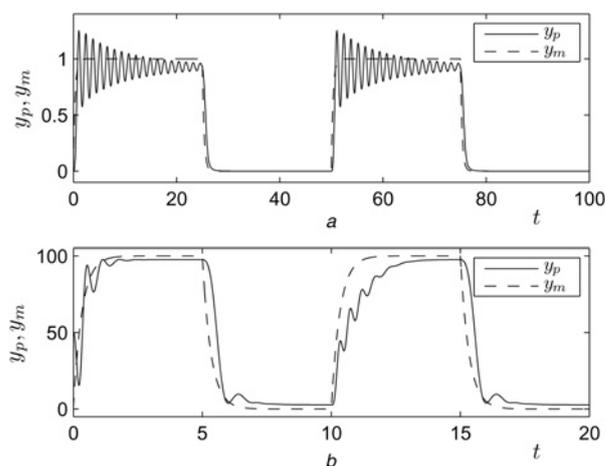
**Figure 4** Performance of the  $\sigma$ -modification with static normalisation

- a Simple conditions
- b Difficult conditions



**Figure 5** Performance of the proposed approach

- a Simple conditions
- b Difficult conditions



**Figure 6** Performance of the  $\sigma$ -modification with static normalisation after correcting  $\Gamma$

- a Simple conditions
- b Difficult conditions

which was done in the experiment with  $\Gamma = 10I$  whose results are shown in Fig. 6. In this case a much better performance of the system in the difficult conditions was achieved (Fig. 6b), whereas the results in the simple conditions were much worse (Fig. 6a).

## 5 Guidelines for choosing the design parameters in direct MRAC

The transfer function  $\tilde{H}_e^a(s)$  that was analysed in the preceding sections will play the central role in this section. Especially important is the frequency response  $\tilde{H}_e^a(j\omega)$ . It should not be necessary to emphasise that the supremum of  $|\tilde{H}_e^a(j\omega)|$  should be as low as possible to approach the ideal

case where  $H_e(s)$  is SPR. In view of Fig. 1 this can be achieved by fulfilling the following condition

$$\omega_{\text{par}} \gg \omega_{\text{ref}} \quad (52)$$

that can be interpreted in the following way: Since in the real systems it is not possible to ensure that  $H_e(s)$  is SPR, it is good for the robustness to come close to this requirement. This is done by selecting a reasonably high bandwidth for the reference model – in any case lower than the dominant part in the unmodelled dynamics. This demand is in accordance with robust linear control and is intuitively clear – when one wants good performance from the system (high bandwidth), it is necessary to cope with reduced robustness.

One of the requirements of Theorems 1 and 2 is that  $\sigma'$  has to satisfy the following inequality

$$\sigma' > -\inf_{\omega \in \mathbb{R}} \Re\{\tilde{H}_e^a(j\omega)\} \quad (53)$$

If the design parameter  $\sigma'$  satisfies the following inequality

$$|\tilde{H}_e^a(j\omega)| < \sigma', \quad \forall \omega \in \mathbb{R} \quad (54)$$

then the inequality (53) will definitely be satisfied. The parameter  $\sigma'$  will not be chosen based on the estimation of  $(-\inf_{\omega} \Re\{\tilde{H}_e^a(j\omega)\})$ , but rather based on the estimation of  $\sup_{\omega} |\tilde{H}_e^a(j\omega)|$ . The estimation based on  $\sup_{\omega} |\tilde{H}_e^a(j\omega)|$  provides a higher value for  $\sigma'$ , although the two estimates are similar. The exact values of  $(-\inf_{\omega} \Re\{\tilde{H}_e^a(j\omega)\})$  and  $\sup_{\omega} |\tilde{H}_e^a(j\omega)|$  for the case treated in Example 1 are 0.2361 and 0.2369, respectively.

From inequality (54) it follows that the estimate of  $\|\tilde{H}_e^a\|_{\infty}$  is needed for the choice of  $\sigma'$ . It is obvious that the estimate of  $\sigma'$  according to inequality (54) is different from the one in (26), but on the other hand unmodelled dynamics are not known (even the dominant dynamics are unknown in adaptive control), and we need a very raw estimation. The next question is: How is  $\|\tilde{H}_e^a\|_{\infty}$  related to  $\omega_{\text{par}}$  and  $\omega_{\text{ref}}$ ? We have three possibilities:

- $\omega_{\text{ref}} > \omega_{\text{par}}$ : the frequency response of  $|\tilde{H}_e^a(\omega)|$  has the shape as shown in Fig. 1b; the leakage parameter that fulfils inequality (54) is too large for the system to have acceptable performance; for these reasons, choosing the reference model bandwidth so high is not preferable.
- $\omega_{\text{ref}} \simeq \omega_{\text{par}}$ : the frequency response of  $|\tilde{H}_e^a(\omega)|$  has the shape as shown in Fig. 1a with the supremum around 0 dB;  $\sigma'$  should be around 1 so that the inequality (54) is fulfilled.
- $\omega_{\text{ref}} < \omega_{\text{par}}$ : this is the preferred choice of the reference model bandwidth that results in a system with acceptable robustness and performance.

When  $\omega_{\text{ref}} < \omega_{\text{par}}$ , the graphical analysis of the diagram in Fig. 1a gives

$$\sigma' = \sqrt{\frac{\omega_{\text{ref}}}{\omega_{\text{par}}}} \quad (55)$$

Since the frequency response is rounded off, the estimate in (55) is a little conservative and results in a value for  $\sigma'$  that is too high. Usually, a choice of  $\sigma'$  between 0.1 and 1 is a good idea. When more robustness is desired, the parameter is increased, while in the case of a high-performance demand the value can be decreased especially when the condition  $\omega_{\text{par}} \gg \omega_{\text{ref}}$  is fulfilled with a high probability.

## 6 Conclusions

An analysis of the direct MRAC with parasitics has shown that the hyperstability of the system can be ensured by adding an extra term to the adaptive law. This term can be interpreted as the leakage in the adaptive law. The algorithm is further developed so that the global stability in the presence of disturbances is proven. The proposed adaptive law has signal-dependent leakage, and is therefore also suitable for controlling systems where the amplitudes of signals change drastically. It has to be pointed out that this is achieved without using signal normalisation. It is shown that under some circumstances a version of the proposed adaptive law is similar to the  $\sigma$ -modification with the static normalisation of signals. For this reason the two approaches are compared in the sense of how they are able to control the famous Rohrs' plant. It turned out that the proposed approach gives better results, especially in the case of very high or very low signals in the regressor  $\psi$ .

A great problem in the adaptive control design is the choice of several design parameters. Since it is shown that the design parameter  $\sigma'$  of the proposed adaptive law is directly related to the  $H_{\infty}$  norm of the parasitic dynamics, the criteria for system stability are derived. Based on these, some guidelines for choosing the leakage parameter  $\sigma'$  and the bandwidth of the reference model are also presented.

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